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SUMMABILITY OF DOUBLE SERIES.

By Lloyd L. Smail.

Practically nothing has as yet been done in the application of methods of summation of divergent series to double series. C. N. Moore* has discussed certain aspects of the Cesàro summability of double series, with the object in view of applications to double Fourier's series. Bromwich and Hardy† have given an extension of Abel's theorem on the continuity of power series to double series summable by an extension of Hölder's method.

The object of this paper is to give a general theorem on a method of summation of double series analogous to the general method of summation for simple series which I gave in my paper on "A General Method of Summation of Divergent Series" in the Annals of Mathematics for December, 1918.

Let $f_{i,j}(m, n, x, y)$ be a function defined for all positive integral and zero values of i and j, and for all real values of m, n, x, y, and satisfying the following conditions:

1°. When m, n, x, y are fixed, for every i, j,

$$(a) \qquad f_{i,j} > 0,$$

$$(b) \qquad h_{i,j} \equiv f_{i,j} - f_{i,j+1} > 0,$$

$$(c) \qquad h'_{i,j} \equiv f_{i,j} - f_{i+1,j} > 0,$$

$$(d) \qquad H_{i,j} \equiv f_{i,j} - f_{i+1,j} - f_{i,j+1} + f_{i+1,j+1} > 0;$$

2°. $\underset{x,y}{\mathrm{L}}\ \underset{m,n}{\mathrm{L}} f_{i,j}(m, n, x, y) = 1$* for i, j fixed;

* By this notation $\underset{x,y}{\mathrm{L}}\ \underset{m,n}{\mathrm{L}}(\ )$, we mean $\underset{x,y}{\mathrm{L}}\{\underset{m,n}{\mathrm{L}}(\ )\}$.

3°. $\underset{x,y}{\mathrm{L}}\ \underset{m,n}{\mathrm{L}} f_{i,n}(m, n, x, y) = 0$ for i, j fixed;

4°. $\underset{x,y}{\mathrm{L}}\ \underset{m,n}{\mathrm{L}} f_{m,j}(m, n, x, y) = 0$ for i, j fixed;

5°. $\underset{x,y}{\mathrm{L}}\ \underset{m,n}{\mathrm{L}} f_{m,n}(m, n, x, y) = 0.$

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* C. N. Moore, "On Convergence Factors in Double Series and the Double Fourier's Series," Trans. Am. Math. Soc., vol. XIV, p. 73.

C. N. Moore, "On the Summability of the Double Fourier's Series of Discontinuous Functions," Math. Ann., vol. LXXIV, p. 555.

† Bromwich and Hardy, "Some Extensions to Multiple Series of Abel's Theorem on the Continuity of Power Series," Proc. London Math. Soc., vol. II, p. 161.

Let $\sum_{m,n} a_{m,n}$ $(m, n = 0, 1, 2, \cdots)$ be any given double series. If we form the expression $\sum_{i=0}^{m}\sum_{j=0}^{n} a_{i,j}f_{i,j}$, and the limit

$$\text{(I)} \qquad \underset{x,y}{\mathrm{L}}\ \underset{m,n}{\mathrm{L}} \sum_{i=0}^{m}\sum_{j=0}^{n} a_{i,j}f_{i,j}\,(m, n, x, y) = S$$

exists, with a finite value S, we shall say that the double series $\sum_{m,n} a_{m,n}$ is *summable* by the summation-function $f_{i,j}\,(m, n, x, y)$, and that S is its *Sum* (or generalized sum).

One of the first problems in a general study of such a method of summation is to determine whether a convergent series is always summable by the method.

As our definition of convergency of a double series we shall take the following: If $S_{m,n} = \sum_{i=0}^{m}\sum_{j=0}^{n} a_{i,j}$ approaches a definite limit S as m and n increase indefinitely, simultaneously but independently, and if for every m, n, we have

$$\text{(1)} \qquad |S_{m,n}| < C \text{ (a positive constant)},$$

then $\sum_{m,n} a_{m,n}$ is convergent with sum S.*

THEOREM. *If the double series* $\sum_{m,n} a_{m,n}$ *is convergent with sum* S, *according to the above definition, then the series will also be summable by definition (I), with the summation-function* $f_{i,j}$, *with generalized sum* S, *provided* $f_{i,j}$ *satisfies the conditions* 1°–5° *above.*

For since $\sum_{i,j} a_{i,j}$ is convergent, we may write

$$\text{(2)} \qquad S_{i,j} = S + \delta_{i,j}, \qquad |\delta_{i,j}| < \epsilon, \qquad i, j \geqq M,$$

where ϵ is any arbitrarily small positive number.

Then by the extension of Abel's identity or partial summation formula for double series,† we have

$$\sum_{i=0}^{m}\sum_{j=0}^{n} a_{i,j}f_{i,j} = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} S_{i,j}H_{i,j} + \sum_{j=0}^{n-1} S_{m,j}h_{m,j} + \sum_{i=0}^{m-1} S_{i,n}h'_{i,n} + S_{m,n}f_{m,n}.‡$$

Taking $m, n > M$, and for $i, j \geqq M$ substituting $S_{i,j} = S + \delta_{i,j}$, we get

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[*] Some writers omit the condition (1) in their definition of convergence, and regard this condition, called the "condition of finitude," as a restriction on convergency. See references to Bromwich and Hardy, and Moore above; also Bromwich, "Infinite Series," §§ 29, 37.

[†] Hardy, Proc. London Math. Soc., Vol. I, p. 124.

[‡] Condition 1° is required for the application of Abel's identity.

$$\begin{aligned}\sum_{i=0}^{m}\sum_{j=0}^{n} a_{i,j}f_{i,j} = \Bigg(&\sum_{i=0}^{M-1}\sum_{j=0}^{M-1} S_{i,j}H_{i,j} + \sum_{i=0}^{M-1}\sum_{j=M}^{n-1} S_{i,j}H_{i,j} + \sum_{i=M}^{m-1}\sum_{j=0}^{M-1} S_{i,j}H_{i,j} \\ &+ \sum_{j=0}^{M-1} S_{m,j}h_{m,j} + \sum_{i=0}^{M-1} S_{i,n}h'_{i,n}\Bigg) + S\Bigg(\sum_{i=M}^{m-1}\sum_{j=M}^{n-1} H_{i,j} + \sum_{j=M}^{n-1} h_{m,j} + \sum_{i=M}^{m-1} h'_{i,n}\Bigg) \\ &+ \Bigg(\sum_{i=M}^{m-1}\sum_{j=M}^{n-1} \delta_{i,j}H_{i,j} + \sum_{j=M}^{n-1} \delta_{m,j}h_{m,j} + \sum_{i=M}^{m-1} \delta_{i,n}h'_{i,n}\Bigg) + S_{m,n}f_{m,n}.\end{aligned} \tag{3}$$

If we keep M fixed, and take the double limit $\underset{x,y}{\mathrm{L}}\ \underset{m,n}{\mathrm{L}}\,(\ )$ on both sides of this equation, we find, by making use of conditions 2°–5° and relations (1) and (2), that

$$\underset{x,y}{\mathrm{L}}\ \underset{m,n}{\mathrm{L}} \sum_{i=0}^{m}\sum_{j=0}^{n} a_{i,j}f_{i,j}\,(m, n, x, y) = S.$$

Thus our general theorem is proved.

Now suppose that we take for $f_{i,j}$ the special form

$$f_{i,j}(m, n, x, y) = f_i(m, x)\cdot f_j(n, y), \tag{4}$$

where $f_i(m, x)$ satisfies the conditions:

(α) $f_i(m, x) > 0$ and $f_i - f_{i+1} > 0$ for every i, m;

(β) $\underset{x}{\mathrm{L}}\ \underset{m}{\mathrm{L}}\, f_i(m, x) = 1$ for i fixed;

(γ) $\underset{x}{\mathrm{L}}\ \underset{m}{\mathrm{L}}\, f_m(m, x) = 0$.

Then it can be shown without any difficulty that this function $f_{i,j}(m, n, x, y) = f_i(m, x)\cdot f_j(n, y)$ satisfies the conditions 1°–5° preceding.

The familiar methods of Cesàro, Hölder, Borel, LeRoy, Riesz, de la Vallée-Poussin, Plancherel, etc., all satisfy the conditions (α), (β), (γ),* so that the summation-function of any one of these well-known methods can be used to build up the summation-function for the summability of double series. If we take the Cesàro function, our general theorem gives as a special case the result of Moore (loc. cit., p. 81).

UNIVERSITY OF WASHINGTON,
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*See Annals of Mathematics, Dec., 1918, p. 154; also my Columbia Dissertation, 1913.